From Allan Variance to Phase Noise: A New Conversion Approach

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Abstract: A new mathematical method is proposed to convert the oscillator instability parameters from Allan variance to Spectrum Density (SD) of random phase fluctuations, which is the inversion of the classic transformation formula from SD to Allan variance. Due to the fact that Allan variance does not always determine a unique SD function, Power-law model of the SD of oscillator phase fluctuations is introduced to the translating algorithm. And that a constrained maximum likelihood solution is presented. Considering that the inversion is an ill-posed problem, a regularization method is brought forward in the process. Simulation results show that the converted SD of phase fluctuations from Allan variance parameters agrees well with the real SD function. Furthermore, the effects of the selected regularization factors and the input Allan variances are analyzed in detail.

Key words: Oscillator instability; Allan Variance; Phase noise; Spectrum Density of Phase Fluctuation; Regularization.

INTRODUCTION

The characterizations of oscillator instabilities can be approached from two points of view: the time domain with Allan variance and the frequency domain with spectrum density of phase fluctuation or phase noise. Barnes deduced the relationships between the two approaches in [1], which gave a theoretical foundation of translation between frequency and time domains. However, the classical equation only provides a one-way translation from phase noise to Allan variance. This translation is used in some stability measurement instruments to give a fast evaluation of oscillator time domain performance from its frequency measures. Barnes also said that for general time domain measures, there is no simple direct expression available for translation them into frequency domain. And that is why one might prefer frequency domain measures as general measures of frequency stability. For some oscillators with specific noise model, a reference table in the appendix II of [1] may be used to translate time and frequency domain measures reciprocally. For other circumstances, it's very difficult to get the frequency domain measures from time domain in general. The reason is that we can't separate the white-PM (Phase Modulation) noise and flicker-PM noise component in time domain.

This paper presents a new translation method from Allan variance to phase fluctuation SD, which is a parametric inversion of classical Barnes formula. The method gives the maximum likelihood (ML) estimation of the different noise PM coefficients in frequency domain from the measured Allan variance. In order to remove the ill-posed process in the inversion, Tikhonov regularization method is used. Computation results show the proposed approach provides a quite good solution for phase fluctuation SD translating from Allan variance.

MATHEMATICAL MODEL OF THE PROPOSED CONVERSION APPROACH

In generally, Allan variance is the often used method for reducing a oscillator noise time series to a statistical summary of frequency stability. Because of Allan variance $\sigma_y^2(\tau)$ acts as an approximate highpass spectral analyzer on PM noise and an approximate octave bandpass analyzer on FM noise. There is a very famous equation, which maps the PM noise SD to Allan variance directly [5~7]:

$$\sigma_y^2(\tau) = \frac{8}{(2\pi f_0 \tau)^2} \int_0^\infty S_\varphi(f) \sin^4(\pi f \tau)\, df \tag{1}$$

where τ is the averaging time, $S_\varphi(f)$ is the phase noise SD, f_0 is the oscillator's nominal frequency. It's quite easy to calculate Allan variance from the phase fluctuation SD using the previous equation. However, for sometimes, we only know the time domain stability measures of one oscillator from the specification index. So how can we get the phase noise SD from the time domain measures such as Allan variances? It's in fact an inverse problem of the (1). For example, there is an oscillator with Allan variance given as $\Theta(\tau)$. It's quite reasonable to assume the given Allan variance, which is often measured directly by the manufacturer, is composed of real Allan variance $\sigma_y^2(\tau)$ of the oscillator and measurement error $n(\tau)$. So we get

$$\Theta(\tau) = \sigma_y^2(\tau) + n(\tau) \tag{2}$$

In (1), the real Allan variance $\sigma_y^2(\tau)$ is dependent on the phase noise of SD. So, in general, the measured Allan variance $\Theta(\tau)$ can be written as

$$\Theta(\tau) = \frac{8}{(2\pi f_0 \tau)^2} \int_0^\infty S_\varphi(f) \sin^4(\pi f \tau)\, df + n(\tau) \tag{3}$$

Let $\Psi(\tau, S_\varphi) = \frac{8}{(2\pi f_0 \tau)^2} \int_0^\infty S_\varphi(f) \sin^4(\pi f \tau)\, df$, we get the following observation equation

$$\Theta(\tau) = \Psi(\tau, S_\varphi) + n(\tau) \tag{4}$$

where, $\Theta(\tau)$ is the observation data, S_φ is the parameter to be estimated, $\Psi(\cdot,\cdot)$ is the kernel function of the inverse problem, and $n(\tau)$ is the observation noise, which is assumed to be Gauss distribution. Then maximum likelihood estimation is given as [8]

$$S_\varphi^{ML}(f) = \arg\min_{S_\varphi} \left\{ \left\| \Theta(\tau) - \Psi(\tau, S_\varphi) \right\| \right\} \tag{5}$$

It's very difficult to solve (5) directly. In fact, Greenhall[9] has already proven that, for an oscillator with given Allan variances, there are more than one SD functions agree with the formula (1). Greenhall points that, some log-period modulated noise components in SD give little contribution to the oscillator's time domain measures. So (5) must be a multi-solution problem. Fortunately, for each oscillator's SD, it is constrained by some power-law forms, and the log-period modulated components disagree with them. Therefore, the power-law model can be used in solving (5) for unique SD function from the measured Allan variances.

CONSTRAINED SOLUTION OF THE SD FUNCTION

As we know, there are mainly about four kinds of noise contributed to oscillator fluctuations, which are thermal noise, shot noise, flicker noise and random walk noise. Among them, the power spectral density of thermal noise and shot noise are flat. So these two kinds of noise are called white noise all together. These kinds of noise are modulated to the oscillator output, which form five components as white PM, flicker PM, white FM, flicker FM and random walk FM. These five kinds of fluctuation component determine that the SD of each oscillator must agree with the power-law form, i.e.

$$S_\varphi(f) = \begin{cases} \mathbf{h}^T\mathbf{F}, or, \sum_{\alpha=-2}^{2} h_\alpha f^{\alpha-2}, & 0 \le f \le f_h \\ 0, & f > f_h \end{cases} \tag{6}$$

where, $\mathbf{h} = (h_{-2}, h_{-1}, h_0, h_1, h_2)^{\mathrm{T}}$, $\mathbf{F} = (f^{-4}, f^{-3}, f^{-2}, f^{-1}, 1)^{\mathrm{T}}$, and " $^{\mathrm{T}}$ "means transpose of matrix. $\alpha = -2, -1, 0, 1, 2$ are the exponents of various noise modulated components, and h_α is the coefficient of the corresponding α . f_h is the upper limit of the measured frequency domain. Substitute (6) into (5), we can get the oscillator S_φ from its time domain measures by estimating coefficients $\mathbf{h} = (h_{-2}, h_{-1}, h_0, h_1, h_2)^{\mathrm{T}}$. In practice, $\Theta(\tau)$ is measured or given as a function of different integral interval τ . Assumed that those discrete intervals are expressed with a vector $\boldsymbol{\tau} = (\tau_1, \cdots, \tau_N)$, the corresponding Allan variances are $\boldsymbol{\Theta} = (\Theta(\tau_1), \cdots, \Theta(\tau_N))$. Then the solution of (5) is equivalent to ML estimation of $\mathbf{h}$ in the following vector equation.

$$\mathbf{h}^{\mathrm{ML}} = \arg\min_{\mathbf{h}} \left\{ \left\| \boldsymbol{\Theta} - \boldsymbol{\Phi}\mathbf{h} \right\| \right\} \tag{7}$$

where, $\boldsymbol{\Phi} = [\boldsymbol{\Psi}(\boldsymbol{\tau}, h_{-2}), \boldsymbol{\Psi}(\boldsymbol{\tau}, h_{-1}), \boldsymbol{\Psi}(\boldsymbol{\tau}, h_0), \boldsymbol{\Psi}(\boldsymbol{\tau}, h_1), \boldsymbol{\Psi}(\boldsymbol{\tau}, h_2)]$ are coefficient matrix related to $\mathbf{h}$, $\boldsymbol{\tau}$, and $\boldsymbol{\Psi}(\boldsymbol{\tau}, h_\alpha) = \left[\Psi(\tau_1, h_\alpha), \cdots, \Psi(\tau_N, h_\alpha)\right]^{\mathrm{T}}$,

$$\boldsymbol{\Psi}(\tau_k, h_\alpha) = \frac{8}{(2\pi f_0 \tau_k)^2}\int_0^\infty f^{\alpha-2}\sin^4(\pi f \tau_k)\,df \qquad k = 1, \cdots, N; \alpha = -2, \cdots, 2$$

Equation (7) can be solved through the generalized inverse matrix theory, i.e.

$$\mathbf{h}^{\mathrm{ML}} = \left(\boldsymbol{\Phi}^{\mathrm{H}}\boldsymbol{\Phi}\right)^{-1}\boldsymbol{\Phi}^{\mathrm{H}}\boldsymbol{\Theta} \tag{8}$$

Where " $^{\mathrm{H}}$ " means conjugate transposing of the matrix. Unfortunately, the matrix $\boldsymbol{\Phi}^{\mathrm{H}}\boldsymbol{\Phi}$ is often ill-conditioned, which means the solution of the equation may deviate seriously from the true value due to some very small measurement errors. To solve this kind of solution unstable problems, Tikhonov regularization is a common method [10], which introduces a regularization function as

$$J_\gamma(\mathbf{h}) = \|\boldsymbol{\Theta} - \boldsymbol{\Phi}\mathbf{h}\|^2 + \gamma\|\mathbf{h}\|^2 \tag{9}$$

The one among the solutions of (9) who gives a minimized $J_\gamma(\mathbf{h})$ will be an approximation of the solution of (8), which has the following expression.

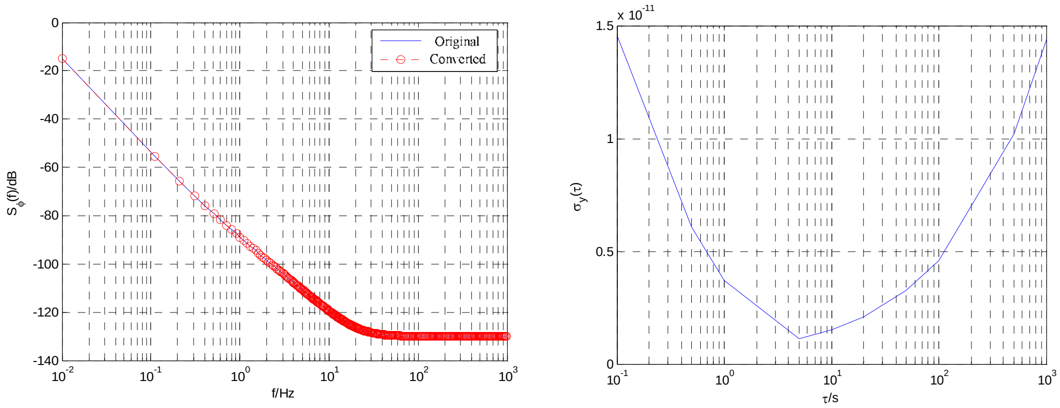
$$\mathbf{h} = \left(\boldsymbol{\Phi}^{\mathrm{H}}\boldsymbol{\Phi} + \gamma\mathbf{I}\right)^{-1}\boldsymbol{\Phi}^{\mathrm{H}}\boldsymbol{\Theta} \tag{10}$$

where $\mathbf{I}$ is unit matrix. In practice, the selection of factor γ is very crucial to the computation. If it is too large, the solution may deviate seriously from the true value. However, if it is small, the solution may still unstable.

SIMULATION VALIDATION AND ANALYSIS

Simulation of the Conversion Approach

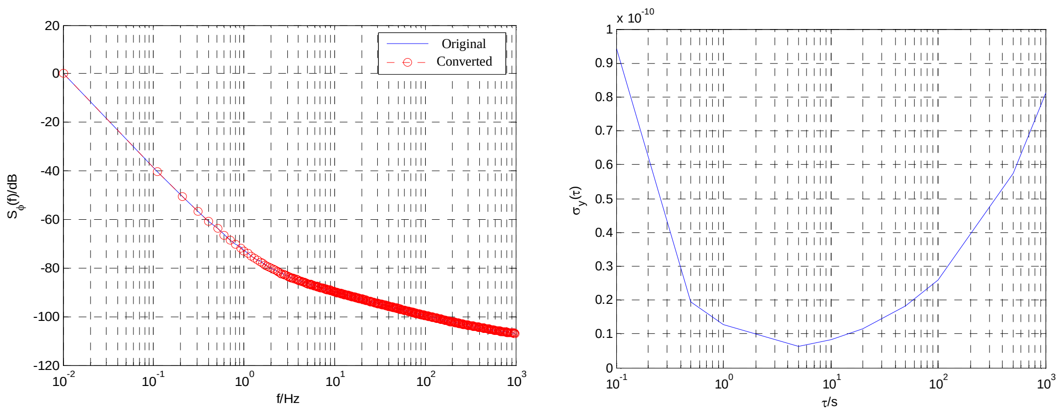
In order to validate the effectiveness of the proposed conversion from Allan variances to SD, two oscillators with corresponding time domain and frequency domain measures are used. First, let's consider the oscillator A, which has a group of coefficients $\mathbf{h} = [10^{-9.5}, 10^{-9}, 10^{-14}, 10^{-13}, 10^{-13}]$ for different power terms. The real line curve in fig. 1(a) gives the SD function of oscillator A. From (1), the Allan variance for various integral intervals is calculated as fig. 1(b). Now let select the $\sigma_y(\tau)$ in fig.1(b) with integral intervals τ =0.1, 0.5, 1.0, 5.0, 10.0, 20, 50.0, 100, 500, 1000 (unit: s) as given measurements. Using the proposed conversion approach, the SD is computed as the curve with"⋯○⋯"in fig. 1(a). Obviously, the converted SD agrees well with the original SD function.



(a) (b)

Fig. 1 Conversion from Allan variance to SD of oscillator A: (a) the comparison of original SD and the converted one, (b) the corresponding Allan variance

For another oscillator B, the power law coefficients are $\mathbf{h} = [10^{-8}, 10^{-7.5}, 10^{-15}, 10^{-12}, 10^{-14}]$. Its SD is the real line curve in fig. 2(a). The corresponding Allan variance is given in fig. 2(b). Choosing the discrete $\sigma_y(\tau)$ value with integral intervals τ =0.1，0.5，1.0，5.0，10.0，20，50.0，100，500，1000 (unit: s) as the measurements, the SD value is gotten using the proposed approach (see "-⊖-" in fig. 2(a)). And the converted SD still agrees well with the original SD curve.



(a) (b)

Fig. 2 Conversion from Allan variance to SD of oscillator B: (a) the comparison of original SD and the converted one, (b) the corresponding Allan variance

In order to evaluate the conversion performance, a quantitative evaluation method is putted forward. Assuming that the original SD of the oscillator is $S_\varphi(f)$, and the converted SD from Allan variance is $S_\varphi^T(f)$, then Mean Translation Error (MTE) and Maximum Translation Error (MxTE) are defined for the general performance evaluation, which have the following expressions.

$$\mathrm{MTE(dB)} = \frac{\int_0^{f_h} abs\left[S_\varphi(f) - S_\varphi^T(f)\right] df}{f_h}$$

$$\mathrm{MxTE(dB)} = \max\left\{abs\left[S_\varphi(f) - S_\varphi^T(f)\right]\right\}$$

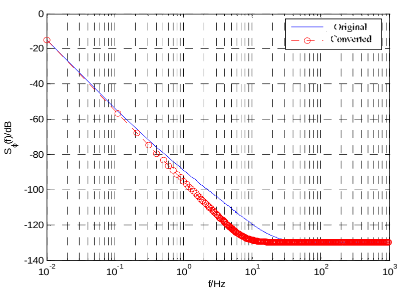
We also define MEF (Maximum Error Frequency) as the frequency at the maximum translation error point for error point evaluation of the conversion. In terms of these index, the two translation performance are listed in Tab. 1 as follow.

Tab. 1 Translation performance of the proposed approach

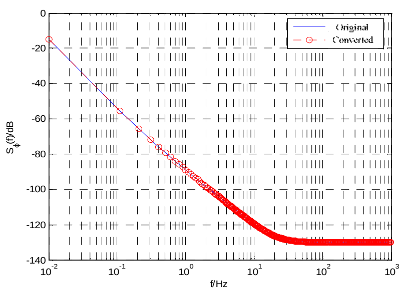
oscillator	Allan variances （$\times 10^{-10}$ ）(τ =0.1，0.5，1.0，5.0，10.0，20，50.0，100，500，1000，unit: s)	MTE(dB)	MxTE (dB)	MEF (Hz)
A	0.1454，0.0609，0.0373，0.0112，0.0149，0.0207，0.0325，0.0458，0.1021，0.1443	2.28×10^{-5}	4.49×10^{-4}	17.1
B	0.9430，0.1933，0.1279，0.0631，0.0842，0.1166，0.1826，0.2574，0.5740，0.8114	2.40×10^{-3}	1.12×10^{-1}	2.0

Selection of the Regularization Factors

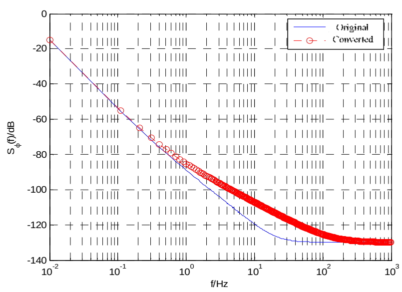
The selection of the regularization factors is very important in the inversion process. Over-regularizated (the factor is larger than ideal value) process makes the inversion quite stable, but the translation error is quite large. Under-regularizated (the factor is smaller than ideal value) process can hardly remove the ill-posed feature. When the measurement is disturbed by some error, the translation will be unstable. In the previous two translations, the condition numbers of matrix $\mathbf{\Phi}^{\mathrm{H}}\mathbf{\Phi}$ are both very large, which mean they are very serious ill-posed problems. In order to estimate the effects of the regularization factors on the translation, different values are selected for translations. Fig. 3 give a group of typical results for oscillator A and B with different γ used in the translation process. For oscillator A, the translation error is very larger when γ equals to 1×10^{-50}. When γ equals to 1×10^{-60}, the converted SD agrees well with the original SD function. When γ is zero, the converted SD deviates from the original one seriously. Tab. 2 list the quantitative performance of the translation for different γ. Obviously, the translation errors are least when γ is 1×10^{-60}. Note that, we will get a better translation when γ is in the vicinity of the minimum eigenvalue of the matrix $\mathbf{\Phi}^{\mathrm{H}}\mathbf{\Phi}$. For oscillator B, the same tests are also done, and the same conclusion is gotten.



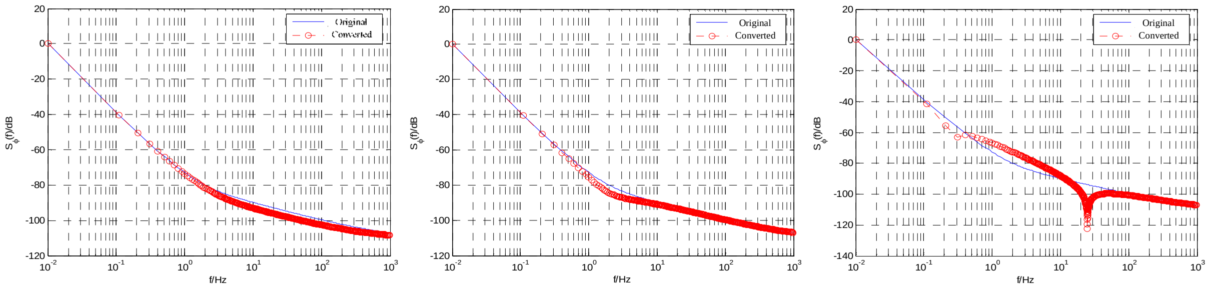
(a)



(b)



(c)



(d) (e) (f)

Fig.3 The effects of regularization factors on SD translations: (a) oscillator A, $\gamma=1\times10^{-50}$, (b) oscillator A, $\gamma=1\times10^{-60}$, (c) oscillator A, $\gamma=0$; (c) oscillator A, $\gamma=1\times10^{-50}$, (d) oscillator A, $\gamma=1\times10^{-70}$, (e) oscillator A, $\gamma=0$

Tab. 2 Quantitative analysis of the effects of regularization factors on SD translation

oscillator	Minimum eigenvalue	Evaluation index	Regularization factor γ			
			1×10^{-50}	1×10^{-60}	1×10^{-70}	0
A	3.3016×10^{-69}	MTE(dB)	0.2167	0.0091	2.1883×10^{-5}	1.4137
		MxTE(dB)	11.4579	0.0579	4.4918×10^{-4}	13.1936
		HEF(Hz)	5.7	27.2	17.1	17.1
B	3.3016×10^{-59}	MTE(dB)	2.1016	0.0024	0.0672	0.8075
		MxTE(dB)	3.3394	0.1118	4.3912	31.8816
		HEF(Hz)	17.1	2.0	2.0	24.9

CONCLUSION

In this paper, a new translation approach from Allan variance to SD of random phase fluctuation is proposed. First, in terms of the classical equation of oscillator stability translation from $S_\varphi(f)$ to $\sigma_y^2(\tau)$, a mathematical model is putted forward which is the inversion of the classical Barnes equation in nature. Due to the fact that Allan variance does not always determine a unique SD function, Power-law model of the SD of oscillator phase fluctuations is introduced to the translating algorithm. And that a constrained maximum likelihood solution is presented. Considering that the inversion is an ill-posed problem, a regularization method is brought forward in the process. Simulation results show that the converted SD of phase fluctuations from Allan variance parameters agrees well with the real SD function. Furthermore, the effects of the selected regularization factors and the input Allan variances are analyzed in detail.

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